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Variational Inequalities and Applications to a Continuum Model of Transportation Network with Capacity Constraints

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Abstract. A continuum model of transportation network is considered in presence of capacity constraints on the flow. The equilibrium conditions are expressed in terms of a Variational Inequality for which an existence theorem is provided.

Key words: Variational inequality, Continuum traffic equilibrium problems, Capacity constraints, Quasi-relative interior, Lagrangean theory

1. Introduction

In ref. [9] the author considers a continuum model of transportation network and characterizes the equilibrium conditions by means of the following Variational Inequality:

$$\text{Find } u \in \mathbb{K} \text{ such that } \int_\Omega c(x,u(x))(v(x)-u(x))\,\mathrm{d}x \geqslant 0 \quad \forall v \in \mathbb{K}, \tag{1}$$

where

$$\mathbb{K} = \{v = \mathcal{E}(\Omega,\mathbb{R}^2) : v_1(x) \geqslant 0,\ v_2(x) \geqslant 0, \operatorname{div} v + t(x) = 0,$$
$$v_1|_{\partial\Omega} = \varphi_1(x),\ v_2|_{\partial\Omega} = \varphi_2(x)\}.$$

Ω is a simply connected bounded domain in $\mathbb{R}^2$ of generic point $x=(x_1,x_2)$, with Lipschitz boundary $\partial\Omega$. $v(x)=(v_1(x),v_2(x))$ represents the unknown flow at each point $x \in \Omega$ and the components $v_1(x)$, $v_2(x)$ are the traffic density through a neighbourhood of x in the directions of the increasing axes x_1 and x_2. $\varphi=(\varphi_1,\varphi_2) \in L^2(\partial\Omega,\mathbb{R}^2)$ is the fixed flow on the boundary $\partial\Omega$ (or on a part of $\partial\Omega$), $c(x,u(x))=(c_1(x,u(x)),c_2(x,u(x)))$ is the 'personal cost' whose components $c_1(x,u(x))$, $c_2(x,u(x))$ represent the travel cost along the axes x_1 and x_2 respectively. It is assumed

(i) $c(x,u) : \Omega \times \mathbb{R}^2 \to \mathbb{R}^2$ is a Carathéodory function such that

$$\|c(x,u)\|_{\mathbb{R}^2} \leqslant \alpha(x) + \|u\|_{\mathbb{R}^2} \text{ a. e. in } \Omega,\ u \in \mathbb{R}^2 \tag{2}$$

with $\alpha(x) \in L^2(\Omega)$.

$\mathcal{E}(\Omega, \mathbb{R}^2)$ is the functional space defined in the following way:

$$\mathcal{E}(\Omega, \mathbb{R}^2) = \{v \in L^2(\Omega, \mathbb{R}^2) : \text{div} \in L^2(\Omega)\}$$

endowed with the norm:

$$\|u\|^2_{\mathcal{E}(\Omega,\mathbb{R}^2)} = \int_\Omega \|u\|^2 \mathrm{d}x + \int_\Omega |\text{div } v|^2 \mathrm{d}x.$$

The equilibrium condition is the following:

DEFINITION 1. $u(x) \in \mathbb{K}$ is an equilibrium distribution flow if there exists a potential $\mu \in H^1(\Omega)$ such that

$$\left(c_i(x, u(x)) - \frac{\partial \mu}{\partial x_i}\right) u_i(x) = 0 \quad i = 1, 2, \text{a. e. in } \Omega \tag{3}$$

$$c_i(x, u(x)) - \frac{\partial \mu}{\partial x_i} \geqslant 0 \quad i = 1, 2, \text{a. e. in } \Omega. \tag{4}$$

μ measures the cost occurred when a network user travels from the point x to the boundary $\partial\Omega$ using the cheapest possible path.

In [9] it is proved the following equivalence result:

THEOREM 1. $u \in \mathbb{K}$ *is an equilibrium distribution according to definition 1 if and only if*

$$\int_\Omega c(x, u(x))(v(x) - u(x)) \mathrm{d}x \geqslant 0 \quad \forall v \in \mathbb{K}.$$

As suggested by some authors (see [5]), the aim of this paper is to consider capacity constraints on the flow:

$$0 \leqslant s(x) \leqslant v(x) \leqslant z(x) \text{a. e. in } \Omega \tag{5}$$

with $s(x) = (s_1(x), s_2(x))$, $z(x) = (z_1(x), z_2(x))$ such that

$$(s_1(x), z_2(x)), (z_1(x), s_2(x)) \in \mathbb{K}. \tag{6}$$

Now the set of feasible flows becomes

$$\tilde{\mathbb{K}} = \{v(x) = \mathcal{E}(\Omega, \mathbb{R}^2) : s(x) \leqslant v(x) \leqslant z(x) \text{a. e. in } \Omega, \text{ div } v(x) + t(x) = 0$$
$$v_1|_{\partial\Omega} = \varphi_1(x), \ v_2|_{\partial\Omega} = \varphi_2(x)\}.$$

In virtue of (6) $\tilde{\mathbb{K}}$ is nonempty and the new formulation of the equilibrium conditions is the following:

DEFINITION 2. $u(x) \in \tilde{\mathbb{K}}$ is an equilibrium distribution flow if there exists a potential $\mu \in H^1(\Omega)$ such that

$$\text{if } s_i(x) < u_i(x) \leqslant z_i(x), \text{ then } c_i(x, u(x)) = \frac{\partial \mu}{\partial x_i}; \tag{7}$$

$$\text{if } u_i(x) = s_i(x), \text{ then } c_i(x, u(x)) \geqslant \frac{\partial \mu}{\partial x_i}; \tag{8}$$

$$\text{if } u_i(x) = z_i(x), \text{ then } c_i(x, u(x)) \leqslant \frac{\partial \mu}{\partial x_i}. \tag{9}$$

Under the assumption (i) we shall prove the following:

THEOREM 2. *$u \in \tilde{\mathbb{K}}$ is an equilibrium distribution according with definition 2 if and only if*

$$\int_\Omega c(x, u(x))(v(x) - u(x))\,\mathrm{d}x \geqslant 0 \quad \forall v(x) \in \tilde{\mathbb{K}}. \tag{10}$$

Moreover we provide an existence result for the Variational Inequality (10). In fact, we shall prove the following

THEOREM 3. *Assume that condition (i) holds and that the following monotonicity condition holds*:

(ii) $\quad \Big(c(x,u) - c(x,v)\Big)(u - v) \geqslant 0 \ \forall u, v \in \tilde{\mathbb{K}}$, *a. e. in* Ω.

Then the Variational Inequality (10) admits solutions.

Theorem 3 ensures the existence of a solution fulfilling the equilibrium conditions (7)–(9). As it is well known, these conditions provide an equilibrium flow that follows the so–called ‘user’s optimization’ approach. This equilibrium flow is different from the equilibrium flow obtained minimizing a cost functional. For more details about these questions we refer to references [4, 5, 7, 8].

2. Proof of Theorem 2.

Let us prove that an equilibrium distribution according to Definition 2 satisfies the Variational Inequality (10). In fact, setting

$$\Omega^i_\mu = \{x \in \Omega : s_i(x) < u_i(x) < z_i(x) \text{ a. e. in } \Omega\}$$

$$\Omega^i_s = \{x \in \Omega : s_i(x) = u_i(x)\} \qquad\qquad i = 1, 2$$

$$\Omega^i_z = \{x \in \Omega : z_i(x) = u_i(x)\},$$

we have:

$$
\begin{aligned}
&\int_\Omega c(x,u(x))(v(x)-u(x))\,\mathrm{d}x\\
&\quad=\int_{\Omega^1_\mu} c_1(x,u(x))(v_1(x)-u_1(x))\,\mathrm{d}x+\int_{\Omega^1_s} c_1(x,u(x))(v_1(x)-u_1(x))\,\mathrm{d}x\\
&\qquad+\int_{\Omega^1_z} c_1(x,u(x))(v_1(x)-u_1(x))\,\mathrm{d}x+\int_{\Omega^2_\mu} c_2(x,u(x))(v_2(x)-u_2(x))\,\mathrm{d}x\\
&\qquad+\int_{\Omega^2_s} c_2(x,u(x))(v_2(x)-u_2(x))\,\mathrm{d}x+\int_{\Omega^2_z} c_2(x,u(x))(v_2(x)-u_2(x))\,\mathrm{d}x\\
&\quad=\int_{\Omega^1_\mu} \frac{\partial\mu}{\partial x_1}(v_1(x)-u_1(x))\,\mathrm{d}x+\int_{\Omega^1_s} c_1(x,u(x))(v_1(x)-s_1(x))\,\mathrm{d}x\\
&\qquad+\int_{\Omega^1_z} c_1(x,u(x))(v_1(x)-z_1(x))\,\mathrm{d}x+\int_{\Omega^2_\mu} \frac{\partial\mu}{\partial x_2}(v_2(x)-u_2(x))\,\mathrm{d}x\\
&\qquad+\int_{\Omega^2_s} c_2(x,u(x))(v_2(x)-s_2(x))\,\mathrm{d}x+\int_{\Omega^2_z} c_2(x,u(x))(v_2(x)-z_2(x))\,\mathrm{d}x\\
&\quad\geqslant\int_{\Omega^1_\mu} \frac{\partial\mu}{\partial x_1}(v_1(x)-u_1(x))\,\mathrm{d}x+\int_{\Omega^1_s} \frac{\partial\mu}{\partial x_1}(v_1(x)-s_1(x))\,\mathrm{d}x\\
&\qquad+\int_{\Omega^1_z} \frac{\partial\mu}{\partial x_1}(v_1(x)-z_1(x))\,\mathrm{d}x+\int_{\Omega^2_\mu} \frac{\partial\mu}{\partial x_2}(v_2(x)-u_2(x))\,\mathrm{d}x\\
&\qquad+\int_{\Omega^2_s} \frac{\partial\mu}{\partial x_2}(v_2(x)-s_2(x))\,\mathrm{d}x+\int_{\Omega^2_z} \frac{\partial\mu}{\partial x_2}(v_2(x)-z_2(x))\,\mathrm{d}x\\
&\quad=\int_{\Omega} \frac{\partial\mu}{\partial x_1}(v_1(x)-u_1(x))\,\mathrm{d}x+\int_{\Omega} \frac{\partial\mu}{\partial x_2}(v_2(x)-u_2(x))\\
&\quad=\int_{\partial\Omega} \mu\Big[(v_1(x)-u_1(x))X_1+(v_2(x)-u_2(x))X_2\Big]\,\mathrm{d}x\\
&\qquad-\int_{\Omega} \mu\Big(\frac{\partial v_1(x)}{\partial x_1}+\frac{\partial v_2(x)}{\partial x_2}-\frac{\partial u_1(x)}{\partial x_1}-\frac{\partial u_2(x)}{\partial x_2}\Big)\,\mathrm{d}x=0,
\end{aligned}
$$

from which the assert follows.

Conversely, let us prove that if $u\in\tilde{\mathbb{K}}$ is a solution to (10) then u fulfills definition 2.

Let us consider the function

$$\psi(v)=\int_\Omega c(x,u(x))(v(x)-u(x))\,\mathrm{d}x \quad v\in\tilde{\mathbb{K}} \tag{11}$$

where $u\in\tilde{\mathbb{K}}$ is a solution to the Variational Inequality (10). The convex set $\tilde{\mathbb{K}}$ satisfies the constraint qualification conditions introduced in [1], namely the 'quasi relative interior of $\tilde{\mathbb{K}}$ non empty' (see also [6]), which replaces the standard Slater condition for the infinite dimensional case. Then, following [3], it is possible to show the following results.

LEMMA 1. *The problem*

$$\min_{v\in\tilde{\mathbb{K}}}\psi(v)(=\psi(u)=0) \tag{12}$$

is equivalent to the problem

$$\begin{aligned}\min_{v\in\mathcal{E}_\varphi(\Omega,\mathbb{R}^2)}\ \sup_{(\mu,\lambda_1,\Lambda_1,\lambda_2,\Lambda_2)\in\mathcal{C}^*}&\Big\{\psi(v)+\int_\Omega\mu(x)\Big(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+t(x)\Big)\mathrm{d}x+\\&-\int_\Omega\lambda_1(x)(v_1(x)-s_1(x))\mathrm{d}x-\int_\Omega\Lambda_1(x)(z_1(x)-v_1(x))\mathrm{d}x+\\&-\int_\Omega\lambda_2(x)(v_2(x)-s_2(x))\mathrm{d}x-\int_\Omega\Lambda_2(x)(z_2(x)-v_2(x))\mathrm{d}x\Big\},\end{aligned} \tag{13}$$

where

$$\mathcal{C}^*=\{(\mu,\lambda_1,\Lambda_1,\lambda_2,\Lambda_2):\mu,\lambda_i,\Lambda_i\in L^2(\Omega),\ \mu,\lambda_i,\Lambda_i \geqslant 0,\ i=,1,2\,a.e.\,in\,\Omega\}$$

and

$$\mathcal{E}_\varphi(\Omega,\mathbb{R}^2)=\{v\in\mathcal{E}(\Omega,\mathbb{R}^2):v_1|_{\partial\Omega}=\varphi_1,\ v_2|_{\partial\Omega}=\varphi_2\}.$$

Let us consider the dual problem:

$$\begin{aligned}\max_{(\mu,\lambda_1,\Lambda_1,\lambda_2,\Lambda_2)\in\mathcal{C}^*}\ \inf_{v\in\mathcal{E}_\varphi}&\Big[\psi(v)+\int_\Omega\mu(x)\Big(\frac{\partial v_1}{\partial x_1}++\frac{\partial v_2}{\partial x_2}+t(x)\Big)\mathrm{d}x+\\&-\int_\Omega\lambda_1(x)(v_1(x)-s_1(x))\mathrm{d}x-\int_\Omega\Lambda_1(x)(z_1(x)-v_1(x))\mathrm{d}x-\\&-\int_\Omega\lambda_2(x)(v_2(x)-s_2(x))\mathrm{d}x-\int_\Omega\Lambda_2(x)(z_2(x)-v_2(x))\mathrm{d}x\Big],\end{aligned} \tag{14}$$

and the problem:

$$\max_{\Lambda\in\Delta}\Lambda \tag{15}$$

where

$$\Delta=\{\Lambda\in\mathbb{R}:\psi(v)+\int_\Omega\mu(x)\Big(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+t(x)\Big)\mathrm{d}x+$$

$$-\int_\Omega\lambda_1(x)(v_1(x)-s_1(x))\mathrm{d}x-\int_\Omega\Lambda_1(x)(z_1(x)-v_1(x))\mathrm{d}x+$$

$$-\int_\Omega\lambda_2(x)(v_2(x)-s_2(x))\mathrm{d}x-\int_\Omega\Lambda_2(x)(z_2(x)-v_2(x))\mathrm{d}x\geqslant$$

$$\geqslant\Lambda\forall v\in\mathcal{E}_\varphi,\ \forall(\mu,\lambda_1,\Lambda_1,\lambda_2,\Lambda_2)\in\mathcal{C}^*\}.$$

The following result holds.

LEMMA 2. $(\overline{\mu},\overline{\lambda}_1,\overline{\Lambda}_1,\overline{\lambda}_2,\overline{\Lambda}_2)\in\mathcal{C}^*$ *is a maximal solution to the dual problem* (14) *if and only if*

$$\overline{\Lambda}=\max_{(\mu,\lambda_1,\Lambda_1,\lambda_2,\Lambda_2)\in\mathcal{C}^*}\inf_{v\in\mathcal{E}_\varphi}\Big[\psi(v)+\int_\Omega\mu(x)\Big(\frac{\partial v_1}{\partial x_1}++\frac{\partial v_2}{\partial x_2}+t(x)\Big)\mathrm{d}x+$$

$$-\int_\Omega\lambda_1(x)(v_1(x)-s_1(x))\mathrm{d}x-\int_\Omega\Lambda_1(x)(z_1(x)-v_1(x))\mathrm{d}x-$$

$$\int_\Omega\lambda_2(x)(v_2(x)-s_2(x))\mathrm{d}x-\int_\Omega\Lambda_2(x)(z_2(x)-v_2(x))\mathrm{d}x\Big]$$

is a solution to (15).

LEMMA 3. *If the primal problem (13)* (*or (12)*) *is solvable, then the dual problem (14) is also solvable and the extremal values of the two problems are equal.*

Now let us consider the Lagrangean function

$$\mathcal{L}:\mathcal{E}_\varphi\times\mathcal{C}^*\to\mathbb{R}$$

defined by setting

$$\mathcal{L}(v,\mu,\lambda_1,\Lambda_1,\lambda_2,\Lambda_2)=\psi(v)+\int_\Omega\mu(x)\Big(\frac{\partial v_1}{\partial x_1}++\frac{\partial v_2}{\partial x_2}+t(x)\Big)\mathrm{d}x+$$

$$-\int_\Omega\lambda_1(x)(v_1(x)-s_1(x))\mathrm{d}x-\int_\Omega\Lambda_1(x)(z_1(x)-v_1(x))\mathrm{d}x-$$

$$-\int_\Omega\lambda_2(x)(v_2(x)-s_2(x))\mathrm{d}x-\int_\Omega\Lambda_2(x)(z_2(x)-v_2(x))\mathrm{d}x.\qquad(16)$$

The following result holds true:

LEMMA 4. *A point* $(u, \overline{\mu}, \overline{\lambda}_1, \overline{\Lambda}_1, \overline{\lambda}_2, \overline{\Lambda}_2) \in \mathcal{E}_\varphi \times \mathcal{C}^*$ *is a saddle point of the Lagrangean function* $\mathcal{L}$, *that is*

$$\mathcal{L}(u, \mu, \lambda_1, \Lambda_1, \lambda_2, \Lambda_2) \leqslant \mathcal{L}(u, \overline{\mu}, \overline{\lambda}_1, \overline{\Lambda}_1, \overline{\lambda}_2, \overline{\Lambda}_2) \leqslant \mathcal{L}(v, \overline{\mu}, \overline{\lambda}_1, \overline{\Lambda}_1, \overline{\lambda}_2, \overline{\Lambda}_2)$$
$$\forall (\mu, \lambda_1, \Lambda_1, \lambda_2, \Lambda_2) \in \mathcal{C}^*, \quad \forall v \in \mathcal{E}_\varphi,$$

if and only if u is a solution o the primal problem, $(\overline{\mu}, \overline{\lambda}_1, \overline{\Lambda}_1, \overline{\lambda}_2, \overline{\Lambda}_2)$ *is a solution to the dual problem* (14) *and the extremal values of the two problems are equal.*

From Lemma 4 it follows that

$$\overline{\mu}(x)\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + t(x)\Big) = 0 \Longleftrightarrow \overline{\mu}(x)t(x) = -\overline{\mu}\Big(\frac{\partial u_1(x)}{\partial x_1} + \frac{\partial u_2(x)}{\partial x_2}\Big) \tag{17}$$

$$\overline{\lambda}_1(x)(u_1(x) - s_1(x)) = 0, \quad \Lambda_1(x)(z_1(x) - u_1(x)) = 0 \text{ a. e. in } \Omega \tag{18}$$

$$\overline{\lambda}_2(x)(u_2(x) - s_2(x)) = 0, \quad \Lambda_2(x)(z_2(x) - u_2(x)) = 0 \text{ a. e. in } \Omega \tag{19}$$

Taking into account (17), (18), (19) and the fact that

$$\min_{v \in \mathcal{E}_\varphi} \mathcal{L}(v, \overline{\mu}, \overline{\lambda}_1, \overline{\Lambda}_1, \overline{\lambda}_2, \overline{\Lambda}_2) = \mathcal{L}(u, \overline{\mu}, \overline{\lambda}_1, \overline{\Lambda}_1, \overline{\lambda}_2, \overline{\Lambda}_2) = 0,$$

using the Gauss formula, we get:*

$$\begin{aligned}
\mathcal{L}(v, \overline{\mu}, \overline{\lambda}_1, \overline{\Lambda}_1, \overline{\lambda}_2, \overline{\Lambda}_2) &= \psi(v) + \int_\Omega \overline{\mu}\Big[\frac{\partial(v_1 - u_1)}{\partial x_1} + \frac{\partial(v_2 - u_2)}{\partial x_2}\Big] \mathrm{d}x + \\
&\quad - \int_\Omega \overline{\lambda}_1(x)[v_1(x) - u_1(x)]\mathrm{d}x + \int_\Omega \overline{\Lambda}_1(x)[v_1(x) - u_1(x)]\mathrm{d}x + \\
&\quad - \int_\Omega \overline{\lambda}_2(x)[v_2(x) - u_2(x)]\mathrm{d}x + \int_\Omega \overline{\Lambda}_2(x)[v_2(x) - u_2(x)]\mathrm{d}x = \\
&= \int_\Omega \Big[c_1(x, u(x)) - \frac{\partial \overline{\mu}}{\partial x_1} - \overline{\lambda}_1 + \overline{\Lambda}_1\Big](v_1 - u_1)\mathrm{d}x + \\
&\quad + \int_\Omega \Big[c_2(x, u(x)) - \frac{\partial \overline{\mu}}{\partial x_2} - \overline{\lambda}_2 + \overline{\Lambda}_2\Big](v_2 - u_2)\mathrm{d}x \geqslant 0 \quad \forall v \in \mathcal{E}_\varphi.
\end{aligned}$$

Then, choosing in turn $v_1 = u_1 + \varphi$, $\forall \varphi \in \mathcal{D}(\Omega)$ and $v_2 = u_2$ and $v_2 = u_2 \pm \varphi$ $\forall \varphi \in \mathcal{D}(\Omega)$ and $v_1 = u_1$, we obtain

$$c_i(x, u(x)) - \frac{\partial \overline{\mu}(x)}{\partial x_i} - \overline{\lambda}_i(x) + \overline{\Lambda}_i(x) = 0 \text{ a. e. in } \Omega \quad i = 1, 2. \tag{20}$$

*$\partial \overline{\mu}/\partial x_i$ are considered in the distribution sense. However, because $c_i - \overline{\lambda}_i + \overline{\Lambda}_i \in L^2(\Omega)$, also $\partial \mu/\partial x_i$ will belong to $L^2(\Omega)$.

From (20), in virtue of (18) and (19), we get, if $s_i(x) \leqslant u_i(x) \leqslant z_i(x)$,

$$\left(c_i(x,u(x)) - \frac{\partial \overline{\mu}(x)}{\partial x_i}\right)(u_i(x) - s_i(x))(z_i(x) - u_i(x)) = 0 \text{ a. e. in } \Omega \quad i = 1,2$$

and hence

$$\left(c_i(x,u(x)) - \frac{\partial \overline{\mu}(x)}{\partial x_i}\right) = 0.$$

If $u_i(x) = s_i(x)$, it results from (18) and (19) $\overline{\Lambda}_i(x) = 0$ and we get

$$c_i(x,u(x)) - \frac{\partial \overline{\mu}}{\partial x_i} = \overline{\lambda}_i \geqslant 0.$$

If $u_i(x) = z_i(x)$ it results from (18) and (19) $\overline{\lambda}_i(x) = 0$ and we get

$$c_i(x,u(x)) - \frac{\partial \overline{\mu}}{\partial x_i} + \overline{\Lambda}_i = 0.$$

Since $\overline{\Lambda}_i(x) \geqslant 0$, it follows $c_i(x,u(x)) \leqslant \frac{\partial \overline{\mu}}{\partial x_i}$. Then Theorem 2 is completely proved. □

3. Proof of Theorem 3

We will prove the existence result taking into account the following classical existence Theorem.

THEOREM 4. *Let E be a real topological vector space and let $\mathbb{K} \subseteq E$ be convex, closed, bounded and nonempty. Let $C : \mathbb{K} \to E^*$ be given such that C is monotone and hemicontinuous along line segments. Then there exists $u \in \mathbb{K}$ such that*

$$\langle C(u), v - u \rangle \geqslant 0 \quad \forall v \in \mathbb{K}.$$

Let us set $C : \tilde{\mathbb{K}} \to \left(\mathcal{E}(\Omega, \mathbb{R}^2)\right)^*$ such that

$$\langle C(u), v \rangle = \int_\Omega c(x,u(x))v(x)\,\mathrm{d}x \quad \forall u \in \tilde{\mathbb{K}}, \quad \forall v \in \mathcal{E}(\Omega, \mathbb{R}^2). \tag{21}$$

From assumption (ii) it follows

$$\int_\Omega (c(x,u(x)) - c(c,v(x)))(u(x) - v(x))\,\mathrm{d}x \geqslant 0$$

and hence the operator c is monotone.

From condition (i), it follows that for each sequence $\lambda_n \to \lambda$, λ_n, $\lambda \in [0,1]$ and $\forall u, v \in \tilde{\mathbb{K}}$ it results

$$\lim_n \int_\Omega \|c(x, \lambda_n u + (1-\lambda_n)v) - c(x, \lambda u + (1-\lambda)v)\|^2 \mathrm{d}x = 0$$

and hence

$$\lim_n \int_\Omega c(x, \lambda_n u + (1-\lambda_n)v)(v-u)\mathrm{d}x = \int_\Omega c(x, \lambda u + (1-\lambda)v)(v-u)\mathrm{d}x,$$

namely the hemicontinuity along line segments.

Since $\tilde{\mathbb{K}}$ is bounded because

$$\|u\|_{\mathcal{E}(\Omega,\mathbb{R}^2)} \leqslant \|z\|_{L^2(\Omega)} + \|t\|_{L^2(\Omega)} \quad \forall u \in \tilde{\mathbb{K}},$$

the proof of Theorem 3 is completed. □

References

1. Borwein, J.W. and Lewis, A.S. Practical Conditions for Fenchel Duality in Infinite Dimensions, in: Théra, M.A. and Baillon, J.B. (eds), *Pitman Research Notes in Mathematics Series* 252, 83–89.
2. Dafermos, S. (1980), Continuum Modelling of Transportation Networks, *Transportation Res.* 14 B, 295–301.
3. Daniele, P. (1999), Lagrangean Function for Dynamic Variational Inequalities, *Rendiconti del Circolo Matematico di Palermo, Serie II* 58, 101–119.
4. Daniele, P. and Maugeri, A. (2000), Vector Variational Inequalities and a Continuum Modelling of Traffic Equilibrium Problem, in Giannessi, F. (ed.), *Vector Variational Inequalities and Vector* Equilibria, Kluwer Academic Publishers, Dordrecht, 97–111.
5. Gwinner, J. (1988), On continuum Modelling of Large Dense Networks in Urban Road Traffic, in Griffiths, J.D. (ed.), *Mathematics in Transport Planning and Control* IMA Conference, Cardiff.
6. Maugeri, A. (1998), Dynamic Models and Generalized Equilibrium Problems, in: Giannessi, F. et al. (eds.), *New Trends in Mathematical Programming*, Kluwer Academic Publishers, Dordrecht, 191–202.
7. Maugeri, A. (1985), New Classes of Variational Inequalities and Applications to Equilibrium Problems, *Methods of Operation Research* 53, 129–131.
8. Maugeri, A. (1987), New Classes of Variational Inequalities and Applications to Equilibrium Problems, *Rendiconti Accademia Nazionale delle Scienze deta dei XL* 11, 224–285.
9. Maugeri, A. (2001), Equilibrium Problems and Variational Inequalities, in: Maugeri, A., Giannessi, F. and Pardalos, P. (eds.), *Equilibrium Problems: Nonsmooth Optimization and Variational* Inequalities Models, Kluwer Academic Publishers, Dordrecht.